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THE KINEMATIC METHOD IN GEOMETRICAL PROBLEMS

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Ю. И. Любич и Л. А. Шор

КИНЕМАТИЧЕСКИЙ МЕТОД В ГЕОМЕТРИЧЕСКИХ ЗАДАЧАХ

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THE KINEMATIC METHOD IN GEOMETRICAL PROBLEMS

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by
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INTRODUCTION

One day while reading a serious mathematical book* we came across a problem which seemed to have got there from the works of Conan Doyle or Stevenson. It dealt with hunting for a hidden treasure. A man was given a map and the following instructions: "At the island go to the gallows (which is represented by the point G in Fig. 1) and from there walk in a straight line to the pine tree (the point P in Fig. 1), measuring the distance. At the pine tree turn through a right angle to the left and walk the same distance in a straight line. Again, from the gallows walk to the oak tree (represented by the point O in Fig. 1), measuring the distance, turn through a right angle to the right and walk the same distance. Join these two end points (represented by the points E_1 and E_2, respectively, in Fig. 1) and you will find the treasure (it is the point T of Fig. 1) at the mid-point."

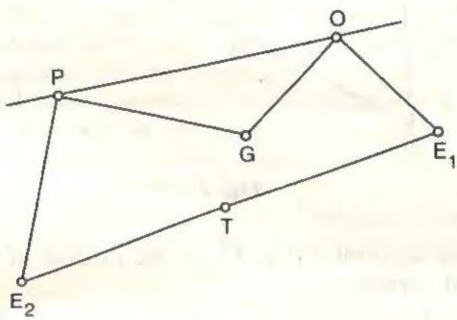


Fig. 1

Finding the treasure with such detailed instructions at one's disposal could present no difficulties. Difficulties did arise, however. When the treasure hunter went to the island, he found that the gallows was gone without trace, but the trees were still there. Nevertheless he found the treasure. How did he manage to do this?

Since the problem was given in a Serious Mathematical book, one should expect that it was not merely a matter of luck. And indeed the problem does have a mathematical solution, which is by the way within the grasp of a high school student.

* Thomas, L. Saaty, Mathematical Methods of Operations Research. New York, etc.: McGraw-Hill, 1959.

Drop perpendiculars from the points E_1, E_2, G and T to the straight line OP (see Fig. 2). Denote their bases by E_1', E_2', G' and T' respectively. Note the congruence of the following pairs of right-angled triangles (by virtue of the equality of their hypotenuses and corresponding acute angles):

$$\triangle OE_1E_1' \equiv \triangle GOG' \quad \triangle PE_2E_2' \equiv \triangle GPG'$$

It follows from the congruence of the triangles that $E_1E_1' = OG'$, $OE_1' = GG'$ and $E_2E_2' = PG'$, $PE_2' = GG'$. Since the point T is the

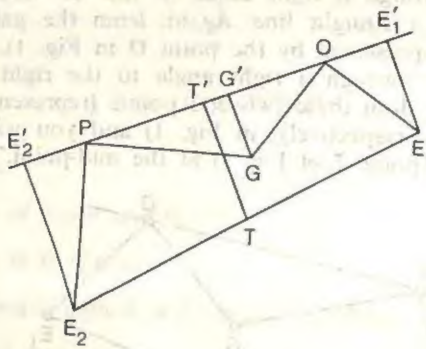


Fig. 2

midpoint of the segment E_1E_2, TT' is the median of the trapezoid $E_1E_1'E_2'E_2$ and therefore

$$TT' = \frac{1}{2}(E_1E_1' + E_2E_2') = \frac{1}{2}(OG' + G'P) = \frac{1}{2}OP$$

Further, the point T' is the midpoint of the segment $E_1'E_2'$ and, since $OE_1' = PE_2'$ $(= GG')$, T' is the midpoint of the segment OP. Thus the position of the point T is independent of the position of the point G. To find the point T, it is sufficient to erect a perpendicular to the segment OP from its midpoint, and to mark off along this perpendicular a segment equal to $\frac{1}{2}OP$ in such a direction that from the point constructed the point O is to the right and the point P to the left.

Although the above solution is faultless, it still leaves something to be desired. The basic idea of dropping perpendiculars from the points E_1, E_2, G and T to the straight line OP is in no way connected with the formulation of the problem and is, in our

opinion, rather artificial.* It is much more natural to find out how the position of the point T depends on the position of the point G or, in other words, how the point T will move if the point G moves. By the way, this idea is suggested by the diagram of the problem. It is easy to imagine that on failing to see the gallows the treasure hunter would begin wandering about in search of its remnants arguing: "If the gallows had been here, then the treasure would be over there, and if it had been here, then...". Then he might notice that the position of the treasure was independent of the position of the gallows. Having noticed this he would start digging, putting off the search for the proof till better times.

Unlike the treasure hunter, we are interested not only in noticing that the position of the point T (the treasure) is independent of the position of the point G (the gallows) by reasoning in this way, but also in proving this to be so.

Imagine that the point G starts moving. Let $\mathbf{v}$ be the vector of its instantaneous velocity. Since the segment OE_1 is obtained from the segment OG by rotating it through an angle $\pi/2$, the point E_1 will move together with the point G, so that the vector $\mathbf{v}_1$ of its velocity will be obtained from the vector $\mathbf{v}$ by a rotation through an angle $\pi/2$. Similarly, the vector $\mathbf{v}_2$ of the velocity of the point E_2 will be obtained from $\mathbf{v}$ by a rotation through an angle** $-\pi/2$. Therefore $\mathbf{v}_2 = -\mathbf{v}_1$. And hence the point T, being the midpoint of the segment E_1E_2, has the velocity

$$\mathbf{u} = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) = 0$$

But if the velocity of a point is always zero, then the point is fixed! So when the point G moves arbitrarily, the point T remains fixed. Consequently, the position of the point T is independent of the position of the point G.

In order to find the position of the point T it is sufficient to choose any position of the point G. Perhaps, the simplest way is to let the point G coincide with the point P and to use the construction known to the treasure hunter (Fig. 3).

---

* Such "artificialities" are very common in the solutions of geometrical problems. This gave the well-known French mathematician J. Favre occasion to say that with many "geometry remains an art of proving some property by considering a craftily chosen circle and successfully joining studiously disconnected points".

** Recall that the angle of rotation is taken to be positive if the rotation is anticlockwise and negative if it is clockwise.

This solution based on kinematic considerations, natural as it is, may seem difficult to a student, not well acquainted with the properties of vectors and velocities.

Therefore, as this little book is devoted to the application of the kinematic method to geometrical problems, we have had to explain quite a lot about vectors and velocities. These concepts play an important role in a number of branches of physics and mathematics. So getting acquainted with them is useful and an end in itself.

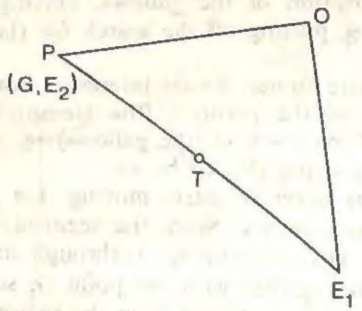


Fig. 3

In Sections 1-2 the results given are mostly not proved but only explained. But by studying the diagrams and reflecting on them, the reader will be able to arrive at sufficiently complete proofs on his own without much difficulty. A reader already familiar with the topics discussed may limit himself to a cursory glance at the material in these sections.

Section 3 is the main section of the book. It shows how to solve a number of problems using the kinematic method, and formulates some problems for independent solution.

1. ELEMENTS OF VECTOR ALGEBRA

1.1. *Vectors* are directed segments. In diagrams vectors are represented by segments with arrowheads indicating their direction (Fig. 4). The initial point of a vector is also called the point of application. A vector with the initial point A and the end point B is denoted by $\overline{AB}$ (but not $\overline{BA}$! $\overline{BA}$ denotes a vector with initial point B and end point A). A single letter is frequently used to denote a vector, for example, $\overline{AB} = a$. It is customary to print this letter in bold type to make it clear at once that

a vector is meant rather than a number. If a vector is denoted by **a**, for example, then its length is denoted by $|\mathbf{a}|$, like the absolute value of a number *. It is also common to call the length of a vector the *magnitude of the vector*.

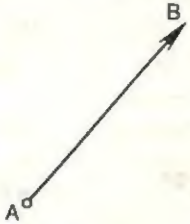


Fig. 4

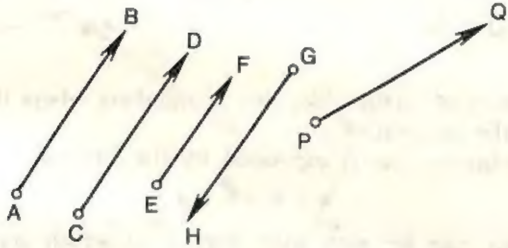


Fig. 5

The equality of two vectors is not understood in the strict sense of complete identity, but in a somewhat broader sense. That is, two vectors are said to be *equal* if they are equal in length and have the same direction. Thus equal vectors are necessarily parallel or lie in the same straight line (more briefly, they are said to be "collinear"). In Fig. 5

$$\overline{AB} = \overline{CD},\ \overline{AB} \neq \overline{PQ}, \qquad \overline{AB} \neq \overline{EF},\ \overline{AB} \neq \overline{GH}$$

It follows from the definition of equality that a vector is not changed by a translation.

It is important for what follows to consider a point as a segment whose initial and end points coincide. Such a "degenerate" segment is also regarded as a vector but it is assigned no definite

* It is common to denote the length of the vector $\overline{AB}$ simply AB.

direction*. It is called the *zero* vector and denoted by $\mathbf{0}$. It has zero length: $|\mathbf{0}| = 0$.

1.2. The *sum* of vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ drawn from the initial point of the vector $\mathbf{a}$ to the end point of the vector $\mathbf{b}$ (Fig. 6), provided that the initial point of the vector $\mathbf{b}$ coincides with the end of the vector $\mathbf{a}$ (this can always be achieved by translating the vector $\mathbf{b}$).

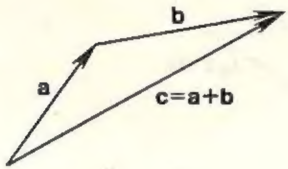


Fig. 6

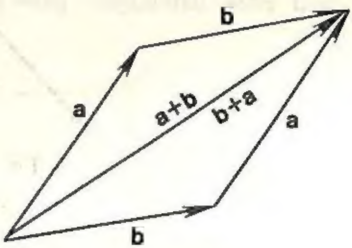


Fig. 7

The addition of vectors, like that of numbers, obeys the commutative and the associative laws.

The *commutative law* is expressed by the formula

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{1}$$

Its validity can be seen from Fig. 7, in which the vectors $\mathbf{a}$ and $\mathbf{b}$ are applied to a point and form the sides of a parallelogram.

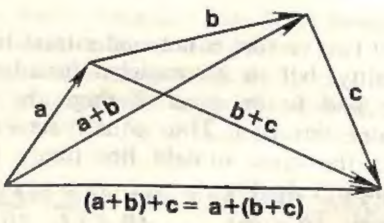


Fig. 8

The diagonal of the parallelogram extending from the common initial point of the vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to the sum $\mathbf{a}+\mathbf{b}$ (as a vector), on one hand, and to the sum $\mathbf{b} + \mathbf{a}$, on the other.

* That is, either direction is regarded as the direction of the vector.

The *associative law* is expressed by the formula

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \tag{2}$$

the validity of which can be seen from Fig. 8.

Due to the commutative and associative laws, it is possible in adding vectors, just as in adding numbers, to disregard both the order and the grouping of the vectors. In particular, it is possible to write simply $\mathbf{a} + \mathbf{b} + \mathbf{c}$, omitting the brackets.

The addition of several vectors is illustrated in Fig. 9 in which the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ applied in succession to one another form an open polygon "closed" by the vector sum $\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4$.

Evidently the sum of several vectors is equal to zero if and only if the broken line they form is closed, i. e. the end of the vector added last coincides with the beginning of the first vector (see Fig. 10).

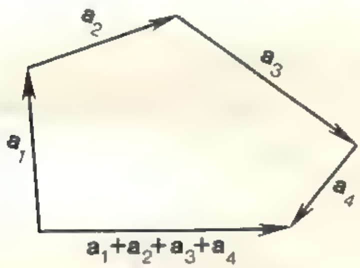


Fig. 9

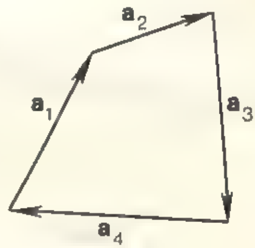


Fig. 10

Let, for example,

$$\mathbf{a} + \mathbf{b} = \mathbf{0}$$

Then the length of the vector $\mathbf{b}$ must be equal to that of the vector $\mathbf{a}$ and its direction exactly opposite to that of the vector $\mathbf{a}$. The vector $\mathbf{b}$ defined in this way is called the *negative* of $\mathbf{a}$ and is denoted by $-\mathbf{a}$.

The formulas

$$\mathbf{a} + \mathbf{0} = \mathbf{a}, \quad \mathbf{a} + (-\mathbf{a}) = \mathbf{0} \tag{3}$$

which follow directly from the definitions play an important part in vector algebra. In particular, they may be used to investigate the operation of subtraction of vectors which is the inverse operation of addition.

1.3. The *difference* $\mathbf{a}-\mathbf{b}$ of the vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector $\mathbf{c}$ such that

$$\mathbf{b}+\mathbf{c}=\mathbf{a} \tag{4}$$

The method for constructing a vector difference is shown in Fig. 11*a*. At the same time it is possible to reduce subtraction to addition as follows. Add to both sides of equation (4) the vector $-\mathbf{b}$:

$$\mathbf{a}+(-\mathbf{b})=(\mathbf{b}+\mathbf{c})+(-\mathbf{b})$$

In virtue of the associative and commutative laws we have

$$\mathbf{a}+(-\mathbf{b})=\mathbf{c}+[\mathbf{b}+(-\mathbf{b})]$$

whence in virtue of formula (3)

$$\mathbf{a}+(-\mathbf{b})=\mathbf{c}+0=\mathbf{c}$$

Thus

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b}) \tag{5}$$

This gives another method for constructing a difference, indicated in Fig. 11*b*.

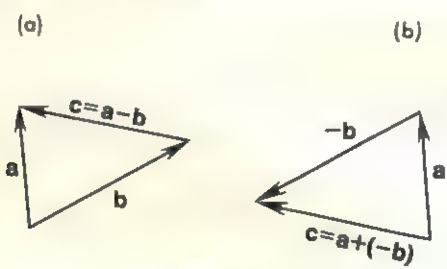


Fig. 11

Note some more formulas:

$$\mathbf{a}-0=\mathbf{a}, \quad 0-\mathbf{a}=-\mathbf{a}, \quad \mathbf{a}-\mathbf{a}=0 \tag{6}$$

By the way, since according to the definition of the difference the equations

$$\mathbf{a}-\mathbf{b}=\mathbf{c}, \quad \mathbf{a}=\mathbf{b}+\mathbf{c}$$

have the same meaning, a vector can be taken from one side of the equation to the other, changing its sign.

1.4. We shall need an important inequality which is called the *triangle inequality*. Turn to Fig. 6. By the well-known geometrical theorem we have the following inequality

$$|\mathbf{a}+\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}| \tag{7}$$

Here the sign of equality is achieved if and only if the vectors have the same direction.

One may cite some more inequalities similar to the triangle inequality; for example,

$$|\mathbf{a}-\mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|; \quad |\mathbf{a}-\mathbf{b}| \geqslant |\mathbf{a}| - |\mathbf{b}| \tag{8}$$

1.5. The *product* $\lambda\mathbf{a}$ of a vector $\mathbf{a}$ and a real number λ is a vector $\mathbf{c}$ given by the following conditions:

(i) $|\mathbf{c}| = |\lambda| \cdot |\mathbf{a}|$ ($|\lambda|$ being the absolute value of the number λ);

(ii) the vectors $\mathbf{c}$ and $\mathbf{a}$ are collinear,

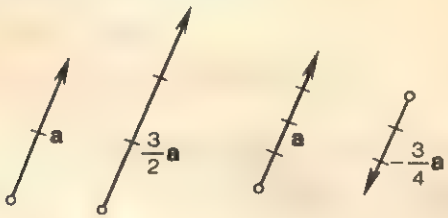


Fig. 12

(iii) for $\lambda > 0$ the direction of the vector $\mathbf{c}$ coincides with that of the vector $\mathbf{a}$, for $\lambda < 0$ these directions are opposite. Fig. 12 presents the cases $\lambda = {}^3/_2$ and $\lambda = -{}^3/_4$. Evidently,

$$\mathbf{a} = 1 \cdot \mathbf{a}, \quad \mathbf{a}+\mathbf{a} = 2\mathbf{a}, \quad \mathbf{a}+\mathbf{a}+\mathbf{a} = 3\mathbf{a}, \ \ldots$$

and

$$-\mathbf{a} = (-1)\cdot\mathbf{a}, \quad (-\mathbf{a})+(-\mathbf{a}) = (-2)\cdot\mathbf{a}$$
$$(-\mathbf{a})+(-\mathbf{a})+(-\mathbf{a}) = (-3)\cdot\mathbf{a}, \ \ldots$$

We shall enumerate the basic laws which govern the multiplication of a vector by a number.

(1) The *associative law*

$$\mu(\lambda\mathbf{a}) = (\mu\lambda)\,\mathbf{a} \tag{9}$$

is illustrated by Fig. 13, which presents the cases $\lambda > 0$, $\mu > 0$ and $\lambda > 0$, $\mu < 0$.

(2) The *distributive law with respect to the numerical multiplier*

$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b} \tag{10}$$

is illustrated by Fig. 14, which presents the cases $\lambda > 0$ and $\lambda < 0$.

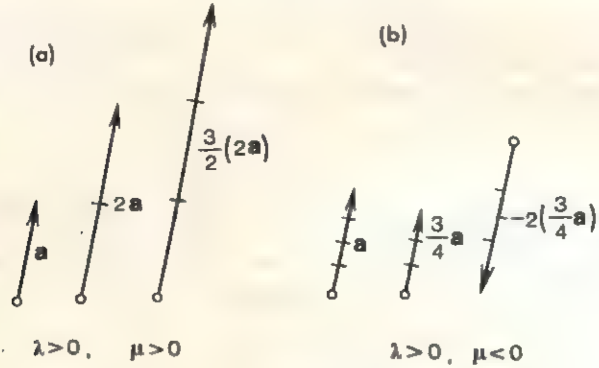


Fig. 13

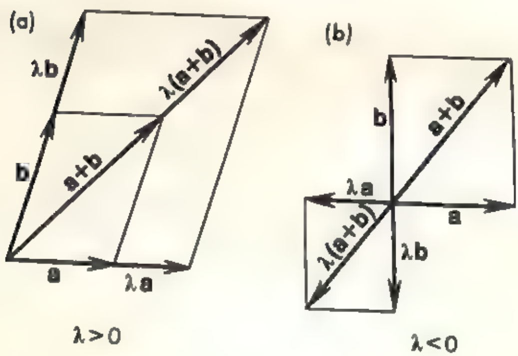


Fig. 14

(3) The *distributive law with respect to the vector multiplier*

$$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a} \tag{11}$$

is illustrated by Fig. 15, which presents the cases $\lambda > 0$, $\mu > 0$ and $\lambda > 0$, $\mu < 0$, $\lambda + \mu > 0$.

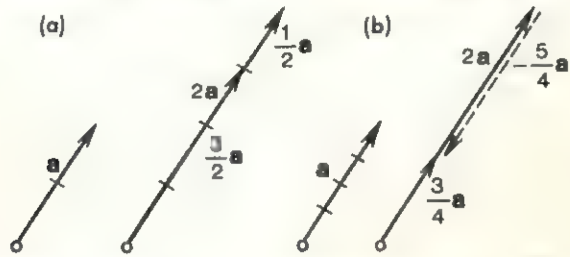
In addition, we want to draw the reader's attention to the obvious identities

$$0 \cdot \mathbf{a} = \mathbf{0}, \quad \lambda \cdot \mathbf{0} = \mathbf{0} \tag{12}$$

It is also possible to introduce the *division of a vector by a number*. The vector obtained by dividing a vector a by a number $\lambda \neq 0$ is the product of the vector a and the number reciprocal to λ:

$$\frac{\mathbf{a}}{\lambda} = \frac{1}{\lambda} \cdot \mathbf{a} \tag{13}$$

So we see that the operations of vector algebra we have considered are governed by the same basic laws as the corresponding operations on numbers. Therefore all the logical consequences of these



$\lambda > 0, \quad \mu > 0$ $\qquad$ $\lambda > 0, \ \mu < 0, \ \lambda + \mu > 0$

Fig. 15

laws hold in vector algebra. This allows one to operate on vectors in the same way as on numbers. For example, it is possible to remove the parentheses in the expression $(\lambda + \mu)(\mathbf{a} + \mathbf{b})$ in the usual way, to give the result $\lambda\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{a} + \mu\mathbf{b}$ (this follows from the distributive laws).

1.6. From now on we shall assume that all vectors lie in a plane*, i. e. we shall be concerned only with plane geometry.

Let **a** and **b** be two noncollinear vectors and **c** be a third vector. If the vector **c** and one of the vectors **a** or **b**, say **a**, are collinear, one can find a number λ such that

$$\mathbf{c} = \lambda\mathbf{a} \tag{14}$$

In the general case, apply all three vectors to one point O (Fig. 16), and after that draw through the end C of the vector **c** straight lines parallel to the vectors **a** and **b**. They will intersect the straight lines on which **a** and **b** lie, in the points A and B respectively.

---

* The reader not acquainted with solid geometry probably assumed so from the very beginning. He has not lost anything essential here.

Clearly,

$$\mathbf{c} = \overline{OA} + \overline{OB}$$

But since the vectors $\overline{OA}$ and $\mathbf{a}$ are collinear, there is a number λ such that

$$\overline{OA} = \lambda\mathbf{a}$$

Similarly there is a number μ such that

$$\overline{OB} = \mu\mathbf{b}$$

Consequently,

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} \tag{15}$$

The representation of the vector $\mathbf{c}$ in the form (15) is called the *decomposition* of the vector into the vectors $\mathbf{a}$ and $\mathbf{b}$. Any vector $\mathbf{c}$ can be decomposed into two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$. Here the coefficients λ and μ are uniquely determined.

Note that the equation (14) may be written in the form (15) when the coefficient $\mu = 0$.

1.7. Let A, B, C be three points lying in a straight line. The point C is said to divide the segment AB in the ratio $m:n$ if*

$$n\overline{AC} = m\overline{CB} \tag{16}$$

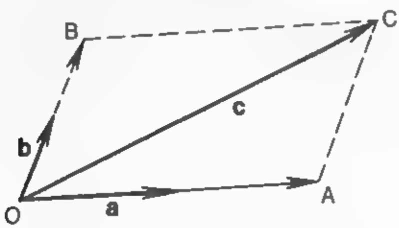


Fig. 16

Evidently, the absolute value of the ratio $m:n$ is equal to the ratio of the lengths $AC:CB$. The ratio $m:n$ is positive if the point C lies on the segment AB and negative if it is outside the segment (Fig. 17).

---

* m, n are any real numbers that are not both simultaneously zero. If $m = 0$, then the point C coincides with the point A; if $n = 0$, then C coincides with B.

THEOREM. *Let the point C divide the segment AB in the ratio $m:n$, and let O be an arbitrary point in the plane* (Fig. 18). *Then*

$$\overline{OC} = \frac{n\overline{OA} + m\overline{OB}}{m+n} \tag{17}$$

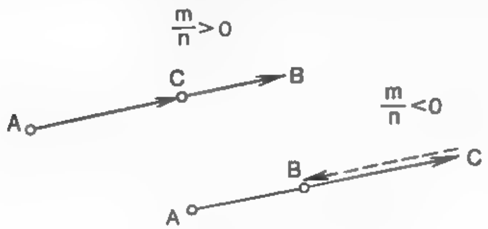


Fig. 17

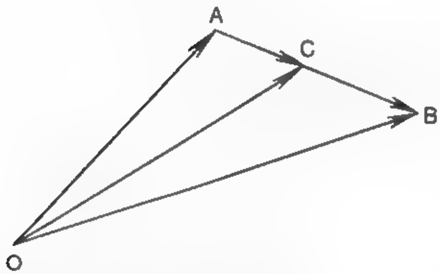


Fig. 18

Conversely, *if for some point O equation* (17) *is true, then the point C divides the segment in the ratio $m:n$.*

PROOF. Let (16) hold. Since

$$\overline{AC} = \overline{OC} - \overline{OA}, \quad \overline{CB} = \overline{OB} - \overline{OC}$$

we have

$$n(\overline{OC} - \overline{OA}) = m(\overline{OB} - \overline{OC})$$

Solving this equation for $\overline{OC}$ we arrive at (17).

Similarly (17) yields (16).

1.8. An *axis* is a straight line provided with a "positive" direction.

Let l be an axis and $\overline{AB}$ a vector (Fig. 19). Denote by A_1 and B_1 the projections of the points A and B on the axis l

(i. e. the feet of the perpendiculars to l drawn through A and B). Consider the number equal to the length of the segments A_1B_1, taken with the plus sign if the direction of the vector $\overline{A_1B_1}$ coincides with that of the axis l, and with the minus sign if it does not. This number is called the *projection* of the vector $\overline{AB}$ on the axis l and denoted by $\mathrm{pr}_l\overline{AB}$.

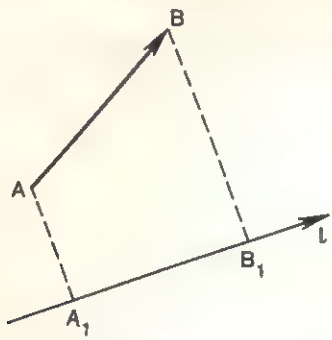


Fig. 19

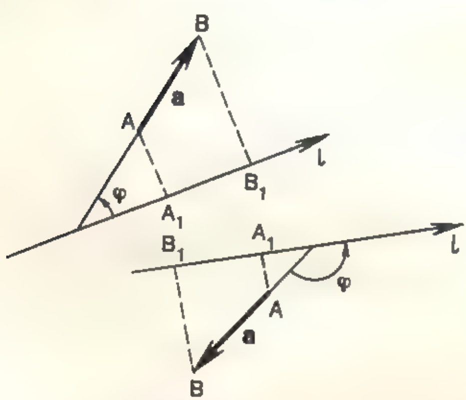


Fig. 20

Let φ be the angle between a vector **a** and an axis l, lying in the interval between 0 and π (Fig. 20). Obviously

$$\mathrm{pr}_l\mathbf{a} = |\mathbf{a}| \cdot \cos\varphi \tag{18}$$

In particular, if **a** is perpendicular to l, then $\mathrm{pr}_l\mathbf{a} = 0$.

Note two more properties of projections (Figs. 21, 22):

(1) $\mathrm{pr}_l(\mathbf{a}+\mathbf{b}) = \mathrm{pr}_l\mathbf{a} + \mathrm{pr}_l\mathbf{b}$,

(2) $\mathrm{pr}_l(\lambda\mathbf{a}) = \lambda\mathrm{pr}_l\mathbf{a}$ (λ being any number).

It is customary to express these properties in the following words: "the projection of a vector on an axis is a *linear operation*

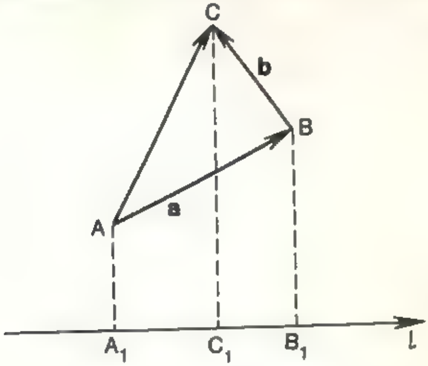


Fig. 21

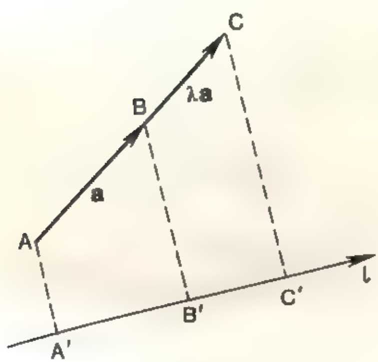


Fig. 22

on vectors". Applying properties (1) and (2) in succession one can write:

$$\mathrm{pr}_l(\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \ldots + \lambda_n\mathbf{a}_n) = \lambda_1\mathrm{pr}_l\mathbf{a}_1 + \lambda_2\mathrm{pr}_l\mathbf{a}_2 + \ldots + \lambda_n\mathrm{pr}_l\mathbf{a}_n \quad (19)$$

for any vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and for any numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Incidentally, multiplication of a vector by a number λ, is also a linear operation (see (9), (10)).

1.9. Another example of a linear operation is given by the operation of rotating a vector through a given angle α (which may be positive, negative or zero). This operation will be denoted by U_α and the result of its application to the vector $\mathbf{a}$ by $U_\alpha\mathbf{a}$. Thus the vector $U_\alpha\mathbf{a}$ is obtained from the vector $\mathbf{a}$ by rotating it through the angle α. Here it is obvious that

$$|U_\alpha\mathbf{a}| = |\mathbf{a}| \tag{20}$$

(Fig. 23).

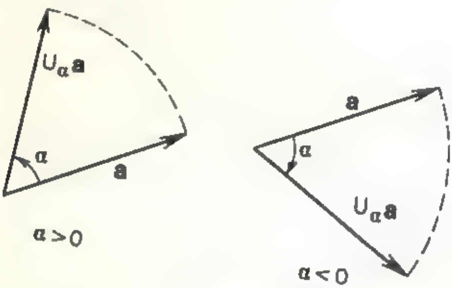


Fig. 23

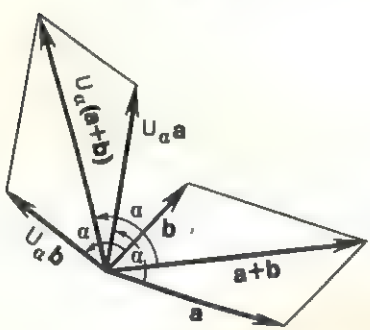


Fig. 24

Evidently, $U_0\mathbf{a} = \mathbf{a}$, i.e. the operation U_0 leaves the vector unchanged. An operation leaving a vector unchanged is called an *identical operation*.

Notice also that

$$U_\pi\mathbf{a} = -\mathbf{a}, \quad U_{2\pi}\mathbf{a} = \mathbf{a} \tag{21}$$

As already stated, the operation of rotation U_α is linear:

(1) $U_\alpha(\mathbf{a}+\mathbf{b}) = U_\alpha\mathbf{a} + U_\alpha\mathbf{b}$ (Fig. 24),

(2) $U_\alpha(\lambda\mathbf{a}) = \lambda U_\alpha\mathbf{a}$, λ being any number (Fig. 25).

Consequently, in a similar manner to (19)

$$U_\alpha(\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \ldots + \lambda_n\mathbf{a}_n) = \lambda_1 U_\alpha\mathbf{a}_1 + \lambda_2 U_\alpha\mathbf{a}_2 + \ldots + \lambda_n U_\alpha\mathbf{a}_n \quad (22)$$

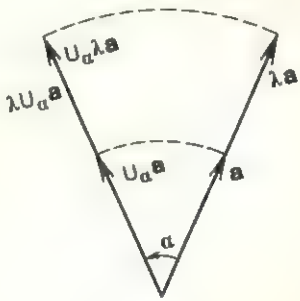


Fig. 25

1.10. Let S, T be two operations on vectors (for example, S is the projection of a vector on some axis l and T is the rotation of the vector through a right angle). The result of performing the two operations in succession is called the *product of the operations*. Note that in general the order in which the operations are performed is important. If in the example just mentioned of the pair of operations S, T we take the vector $\mathbf{a} \neq \mathbf{0}$ on the axis l and apply first T and then S to it, then we get 0. If, however, we first apply S to $\mathbf{a}$, then we get a number, and it is impossible to apply the operation T to a number (as by definition it can be applied only to vectors).

If one first performs the operation T and then the operation S, then the product is written as ST. Thus by definition

$$(ST)\mathbf{a} = S(T\mathbf{a}) \quad (23)$$

for any vector $\mathbf{a}$.

Fig. 26 shows that $U_\beta U_\alpha\mathbf{a} = U_{\alpha+\beta}\mathbf{a}$ for any vector $\mathbf{a}$, i.e.*:

$$U_\beta U_\alpha = U_{\alpha+\beta} \quad (24)$$

---

* Two operations T_1, T_2 on vectors are considered to be equal if $T_1\mathbf{a} = T_2\mathbf{a}$ for all vectors $\mathbf{a}$.

4–632

Hence it can be seen that

$$U_\alpha U_\beta = U_\beta U_\alpha \tag{25}$$

although in general $ST \neq TS$.

If $ST = TS$, then the operations S and T are said to be *commutative*. Thus any two rotations are commutative. The operations

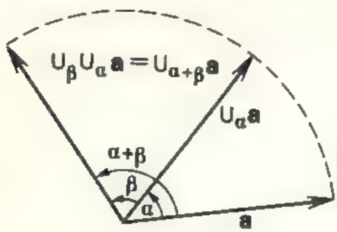


Fig. 26

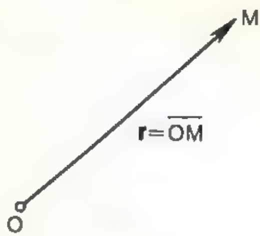


Fig. 27

of rotation and multiplication by a number are also commutative (property (2) of the operation U_α).

It follows from formula (24) that

$$U_{-\alpha} U_\alpha = U_0$$

i. e. $U_{-\alpha} U_\alpha \mathbf{a} = \mathbf{a}$ for any vector $\mathbf{a}$.

The properties of rotations and other geometrical transformations can be used in solving various problems (see I. M. Yaglom, Geometric Transformations. New York, Random House, 1962).

2. ELEMENTS OF KINEMATICS

2.1. Take some point O, as a *pole* in the plane. For an arbitrary point M the vector $r = \overline{OM}$ (Fig. 27) is called the *radius vector* relative to the pole O. The point and its radius vector mutually define each other.

If the point moves along some path (Fig. 28), then its radius vector changes with the time. The radius vector is a *function* of time, and is denoted in the following way:

$$\mathbf{r} = \mathbf{r}(t) \tag{1}$$

where t is the time.

The word "changes" must not be understood too literally here. An important particular case of motion is when the point is at rest. In this case its radius vector will be the same for all points

of time. As a function of time, it is constant. This may be:

$$\mathbf{r} = \text{const} \tag{2}$$

When we say $\mathbf{r}$ is a function of t, we mean that for every value of t the vector $\mathbf{r}$ is completely determined. This means that if t is fixed, then $\mathbf{r}$ is also fixed and can no longer change.

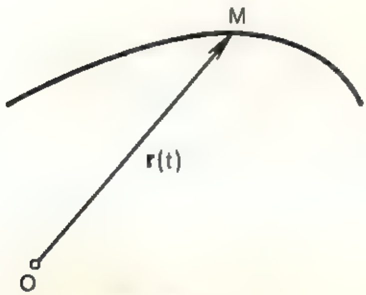


Fig. 28

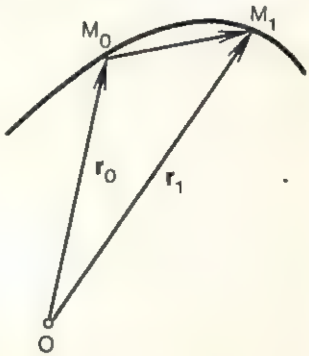


Fig. 29

2.2. Consider some time interval $[t_0, t_1]$ $(t_1 > t_0)$ beginning at time t_0 and ending at time t_1. The duration of the interval is*

$$\Delta t = t_1 - t_0 \tag{3}$$

If at time t_0 the radius vector of a moving point M is $\mathbf{r}_0$ $(\mathbf{r}_0 = \mathbf{r}(t_0))$ and at t_1 it is $\mathbf{r}_1$ $(\mathbf{r}_1 = \mathbf{r}(t_1))$, then the vector

$$\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$$

represents the displacement of the point M in the time interval $[t_0, t_1]$ (Fig. 29).

Now we want to introduce the most important concept of *velocity*. Roughly speaking, velocity is displacement per unit time. But velocity must describe both the absolute magnitude of displacement per unit time and the direction of the displacement, that is, velocity must be a vector.

If we know that in the time interval $[t_0, t_1]$ the point M undergoes a displacement $\Delta\mathbf{r}$, then to obtain the displacement per unit time it is natural to divide $\Delta\mathbf{r}$ by the duration of the time

---

* The symbol Δ is used to denote an *increment* of some value, i. e. to denote how much the value has changed.

interval. The result will be a vector which is called the *average velocity of the point in the given time interval:*

$$\mathbf{v}_{av} = \frac{\Delta \mathbf{r}}{\Delta t} \tag{4}$$

This vector has the same direction as the displacement vector $\Delta \mathbf{r}$, but its magnitude is equal to the distance M_0M_1 divided by Δt, i. e., roughly speaking, to the length of the path travelled by the point per unit time.

Why do we use the words "roughly speaking"? The reason is that as a rule the point M moves nonuniformly in the time interval $[t_0, t_1]$ i. e. it travels unequal distances in equal portions of this time interval. Furthermore, in general it moves not along the straight line M_0M_1, but along a curve joining these points. The displacement vector $\Delta \mathbf{r}$ describes only the result of the movement, but not its intermediate stages. This also applies to the vector of the average velocity, and this is stressed by the word "average".

It is easy to see, however, that the average velocity will describe the motion sufficiently accurately if the duration of the time interval is very small. So in order to obtain an accurate description, we must let the time Δt tend to zero, i. e. fixing

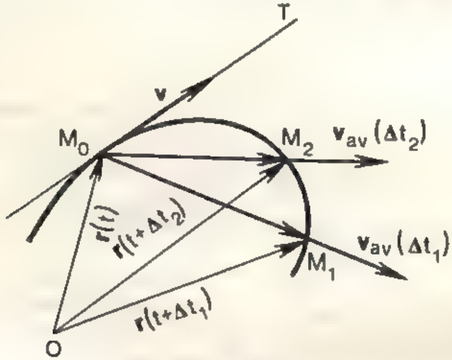


Fig. 30

the beginning of the time interval t_0, we must let t_1 tend to t_0. The average velocity $\mathbf{v}_{av}$ will in general tend to some limit $\mathbf{v}$:

$$\mathbf{v} = \lim_{\Delta t \to 0} \mathbf{v}_{av} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \tag{5}$$

This is to some extent illustrated by Fig. 30.

The vector $\mathbf{v}$ is called the (*instantaneous*) *velocity of the motion at time* t_0. Its direction is the limit of the directions of the average velocity vectors.

The average velocity vector in the time interval $[t_0, t_1]$ lies on the secant M_0M_1. If t_1 tends to t_0, then the point M_1 moving along a path tends to the point M_0. In this case the secant M_0M_1, rotating, tends to some limiting position M_0T. The straight line which is the limit of the secant is called the *tangent to the path at the point* M_0.* The velocity vector at t_0 lies along the tangent to the path at the point M_0.

A reader unfamiliar with these concepts may feel confused by the words "tends", "limit", "limiting position". Furthermore, we have applied these words to variable vectors (and even to variable straight lines), not only to numbers. As for variable numbers, the reader must be familiar with the meaning of those words from a school mathematics course where the elements of the theory of limits are given. But we need a more general theory which we now proceed to outline.

2.3. Let $\rho(s)$ be a function of a numerical argument s which assumes numerical values.** We shall remind the reader of the precise meaning of the equation

$$\lim_{s\to 0} \rho(s) = 0 \tag{6}$$

which is read: "the function $\rho(s)$ tends to zero as s tends to zero". It means that however small the number $\varepsilon > 0$ may be, there can be found a number $\delta > 0$ so small that the inequality

$$|\rho(s)| < \varepsilon$$

holds for all $|s| < \delta$.

Now let $\mathbf{a}(s)$ be a vector function of an argument s. A vector $\mathbf{b}$ is said to be the *limit* of $\mathbf{a}(s)$ as s tends to zero (written $\mathbf{b} = \lim\limits_{s\to 0} \mathbf{a}(s)$), if the scalar function

$$\rho(s) = |\mathbf{a}(s) - \mathbf{b}|$$

tends to zero as s tends to zero.

---

* The reader is recommended to compare this general definition of the tangent with the usual "school-book" definition of a tangent to a circle.

** Such functions are called *scalar* functions to differentiate them from *vector* functions.

The basic theorems of the theory of limits for vector functions are similar to the theorems for scalar functions with which the reader is already familiar.

THEOREM 1. *A function cannot have two different limits*.*

PROOF. Let

$$\mathbf{b}_1 = \lim_{s \to 0} \mathbf{a}(s) \quad \text{and} \quad \mathbf{b}_2 = \lim_{s \to 0} \mathbf{a}(s)$$

Obviously

$$\mathbf{b}_1 - \mathbf{b}_2 = [\mathbf{b}_1 - \mathbf{a}(s)] + [\mathbf{a}(s) - \mathbf{b}_2]$$

So by the triangle inequality,

$$|\mathbf{b}_1 - \mathbf{b}_2| \leqslant |\mathbf{b}_1 - \mathbf{a}(s)| + |\mathbf{a}(s) - \mathbf{b}_2|$$

Both terms on the right-hand side of this inequality vanish as $s \to 0$. Thus, as the left-hand side is independent of s, it cannot be positive:

$$|\mathbf{b}_1 - \mathbf{b}_2| \leqslant 0$$

But it cannot be negative either, as the length of a vector is always nonnegative. Consequently,

$$|\mathbf{b}_1 - \mathbf{b}_2| = 0$$

This means that $\mathbf{b}_1 - \mathbf{b}_2 = \mathbf{0}$, i. e. $\mathbf{b}_1 = \mathbf{b}_2$.

THEOREM 2. *If*

$$\lim_{s \to 0} \mathbf{a}_1(s) = \mathbf{b}_1, \quad \lim_{s \to 0} \mathbf{a}_2(s) = \mathbf{b}_2$$

then

$$\lim_{s \to 0} (\mathbf{a}_1(s) + \mathbf{a}_2(s)) = \mathbf{b}_1 + \mathbf{b}_2$$

("*the limit of a sum is equal to the sum of limits*").

THEOREM 3. *If*

$$\lim_{s \to 0} \mathbf{a}(s) = \mathbf{b}$$

then for any fixed number λ *we have*

$$\lim_{s \to 0} \lambda \mathbf{a}(s) = \lambda \mathbf{b}$$

* The limit may not exist, but if it does, then it is unique.

("*the limit of the product of a number (scalar) and* $\mathbf{a}(s)$ *is equal to the product of the number and the limit of* $\mathbf{a}(s)$).

The proofs of Theorems 2 and 3 are left to the reader *.

THEOREM 4. *If*

$$\lim_{s \to 0} \mathbf{a}(s) = \mathbf{b}$$

then for any fixed angle α *we have*

$$\lim_{s \to 0} U_\alpha \mathbf{a}(s) = U_\alpha \mathbf{b}$$

PROOF. We have (see formula (20) of Sec. 1):

$$|U_\alpha \mathbf{a}(s) - U_\alpha \mathbf{b}| = |U_\alpha (\mathbf{a}(s) - \mathbf{b})| = |\mathbf{a}(s) - \mathbf{b}|$$

Since by assumption

$$\lim_{s \to 0} |\mathbf{a}(s) - \mathbf{b}| = 0$$

we also have

$$\lim_{s \to 0} |U_\alpha \mathbf{a}(s) - U_\alpha \mathbf{b}| = 0$$

i. e.

$$\lim_{s \to 0} \mathbf{U}_\alpha \mathbf{a}(s) = U_\alpha \mathbf{b}$$

2.4. Now we are in a position to present the theory of velocities in the form we need.

The velocity of a point is given by equation (5) **.

The following result is physically obvious.

THEOREM 5. *The velocity of a fixed point is at all times* *** *zero.*

PROOF. Indeed, if a point is fixed, then for any time interval its displacement vector is zero, i. e. $\Delta \mathbf{r} = \mathbf{0}$.

Consequently, $\mathbf{v}_{av} = \Delta \mathbf{r}/\Delta t = \mathbf{0}$. But then $\mathbf{v} = \lim\limits_{\Delta t \to 0} \mathbf{v}_{av} = \mathbf{0}$ at any instant.

---

* When proving Theorem 2 one should make use of the triangle inequality.

** A reader familiar with differentiation may say "the velocity of a point is the derivative of its radius vector with respect to time". Differentiation is one of the most important operations in mathematics. For a presentation of differentiation intelligible to a high school student the reader is referred to V. G. Boltyansky, Differentiation Explained, Mir Publishers, 1977.

*** As long as the point is fixed, of course.

We shall formulate the converse of the theorem.

THEOREM 5′. *If the velocity of a point is all the time zero* (i. e. *for the time during which the motion of the point is considered*) *then the point remains fixed.*

In spite of all its physical obviousness, the theorem is not so simple mathematically. We shall leave out its proof in order not to depart too far from our main topic.

Theorems 5 and 5′ say that the equation

$$\mathbf{r} = \text{const}$$

is equivalent to the equation $v = 0$.

THEOREM 6. *Let* $\mathbf{r}_1 = \mathbf{r}_1(t)$, $\mathbf{r}_2 = \mathbf{r}_2(t)$, $\mathbf{r} = \mathbf{r}(t)$ *be the radius vectors of the points* M_1, M_2, M *respectively. If the points move so that for all points of time*

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$$

then their velocities are connected by a similar relation

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \tag{7}$$

PROOF. The displacement vector of the point M in the time interval $[t, t + \Delta t]$ is

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) = [\mathbf{r}_1(t + \Delta t) + \mathbf{r}_2(t + \Delta t)] - [\mathbf{r}_1(t) + \mathbf{r}_2(t)] = [\mathbf{r}_1(t + \Delta t) - \mathbf{r}_1(t)] + [\mathbf{r}_2(t + \Delta t) - \mathbf{r}_2(t)] = \Delta\mathbf{r}_1 + \Delta\mathbf{r}_2$$

Hence

$$\mathbf{v}_{av} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\Delta\mathbf{r}_1}{\Delta t} + \frac{\Delta\mathbf{r}_2}{\Delta t} = \mathbf{v}_{1av} + \mathbf{v}_{2av}$$

and by Theorem 2

$$\mathbf{v} = \lim_{\Delta t \to 0} \mathbf{v}_{av} = \lim_{\Delta t \to 0} \mathbf{v}_{1av} + \lim_{\Delta t \to 0} \mathbf{v}_{2av} = \mathbf{v}_1 + \mathbf{v}_2$$

which completes the proof.

A similar theorem holds for the difference.

THEOREM 6′. *If the velocities of the points* M_1, M_2, M *are at all times connected by relation* (7), *then at all times we have*

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2 + \text{const} \tag{8}$$

PROOF. Consider an auxiliary point P whose radius vector is at all times equal to

$$\overline{OP} = \mathbf{r} - (\mathbf{r}_1 + \mathbf{r}_2) \tag{9}$$

The velocity of the point P is, according to what has been proved earlier, $\mathbf{v}-(\mathbf{v}_1+\mathbf{v}_2)$, i. e. zero. Consequently, the point P is fixed, i. e. $\overline{OP}=$ const. Hence (9) also immediately yields (8).

It is possible to prove the following pairs of theorems (using Theorems 3, 4) in the same way as Theorems 6 and 6′ were proved.

THEOREM 7. *Let* $\mathbf{r}_1=\mathbf{r}_1(t)$, $\mathbf{r}_2=\mathbf{r}_2(t)$ *be the radius vectors of the points* M_1 *and* M_2 *respectively. If the points move so that we always have*

$$\mathbf{r}_2=\lambda\mathbf{r}_1$$

where λ is a constant number, then their velocities are connected by a similar relation

$$\mathbf{v}_2=\lambda\mathbf{v}_1 \tag{10}$$

THEOREM 7′. *If the velocities of the points* M_1, M_2 *are at all times connected by relation* (10), *then at all times*

$$\mathbf{r}_2=\lambda\mathbf{r}_1+\text{const}$$

THEOREM 8. *Let* $\mathbf{r}_1=\mathbf{r}_1(t)$, $\mathbf{r}_2=\mathbf{r}_2(t)$ *be the radius vectors of the points* M_1 *and* M_2 *respectively. If the points move so that at all times*

$$\mathbf{r}_2=U_\alpha\mathbf{r}_1$$

where α *is a constant angle, then their velocities are connected by a similar relation*

$$\mathbf{v}_2=U_\alpha\mathbf{v}_1 \tag{11}$$

THEOREM 8′. *If the velocities of the points* M_1, M_2 *are at all times connected by relation* (11), *then at all times*

$$\mathbf{r}_2=U_\alpha\mathbf{r}_1+\text{const}$$

2.5. Let $\mathbf{r}=\mathbf{r}(t)$ be the radius vector of a moving point M. Consider the displacement $\overline{M_0M_1}=\Delta\mathbf{r}$ of the point in some time interval $[t_0, t_1]$ (Fig. 31). Draw an arc of a circle with centre at the pole O and radius OM_0. It will meet the ray OM_1 at the point M^*. Evidently,

$$\Delta\mathbf{r}=\overline{M_0M_1}=\overline{M_0M^*}+\overline{M^*M_1} \tag{12}$$

The displacement $\overline{M_0M^*}$ does not alter the distance of the moving point M from the pole O and is due only to the rotation of the ray OM. The displacement $\overline{M^*M_1}$ is due only to the change in the distance of the point M from the pole.

By dividing both sides of equation (12) by $\Delta t = t_1 - t_0$ and proceeding to the limit as $\Delta t \to 0$ we get

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\overline{M_0M^*}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{M^*M_1}}{\Delta t} \tag{13}$$

The first limit on the right-hand side of equation (13) is called the *transversal velocity* of the point M and denoted by $\mathbf{v}_\tau$, the second is called the *radial velocity* of the point M and is denoted by $\mathbf{v}_\rho$.

Hence

$$\mathbf{v} = \mathbf{v}_\tau + \mathbf{v}_\rho \tag{14}$$

Formula (14) gives the decomposition of a vector of velocity into the radial and the transversal component (Fig. 32). These components are mutually perpendicular.

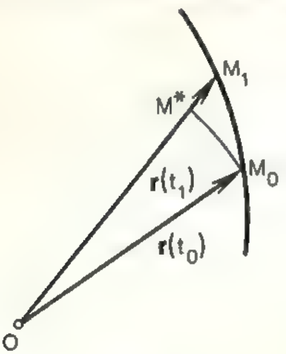


Fig. 31

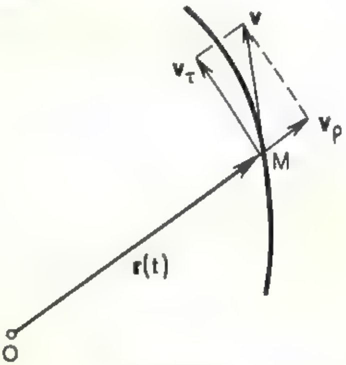


Fig. 32

Radial velocity is the rate of change of the distance of a point M from the pole O or, equivalently, the rate of change of the length of the radius vector $\overline{OM}$. It is directed along the vector if OM increases and in the opposite direction if OM decreases.

Denote the projection of the velocity vector of the point M on the axis given by the vector $\overline{OM}$ by v_ρ. Clearly,

$$v_\rho = \pm |\mathbf{v}_\rho|$$

where the plus sign is taken if OM increases and the minus sign is taken if it decreases.

If the point M moves in a circle with its centre at the pole, then its total velocity coincides with the transversal velocity:

$$\mathbf{v} = \mathbf{v}_\tau, \quad \mathbf{v}_\rho = 0$$

If, however, the point moves along a ray drawn from the pole, then its total velocity coincides with the radial velocity:

$$\mathbf{v} = \mathbf{v}_\rho, \quad \mathbf{v}_\tau = \mathbf{0}$$

2.6. Consider the rotation of a ray OM about its initial point O. Suppose that in the time interval $[t, t + \Delta t]$ the ray rotates through an angle* $\Delta\varphi$. The quotient

$$\omega_{av} = \frac{\Delta\varphi}{\Delta t}$$

is called the *average angular velocity* of the ray in the time interval $[t, t + \Delta t]$. The limit of the average angular velocity ω_{av} as $\Delta t \to 0$ is called the (*instantaneous*) *angular velocity* of the ray and is denoted simply by ω:

$$\omega = \lim_{\Delta t \to 0} \omega_{av} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t}.$$

The angular velocity is not a vector, but a number**. It is positive if the rotation of the ray takes place in the positive direction and negative if the ray rotates in the negative direction.

The following theorems, which are similar to the theorems on the velocities of points, are valid for angular velocities.

THEOREM 9. *The angular velocity of a ray is always zero if and only if the ray is always fixed.*

THEOREM 10. *The angular velocities of two rays*** O_1M *and* O_2M *are at all times equal if and only if the angle between the rays is constant.*

3. THE KINEMATIC METHOD IN GEOMETRICAL PROBLEMS

Now we are equipped to go on to solve geometrical problems. We first of all recommend that the reader re-examine the kinematic solution of the "treasure-hunting problem" discussed in the Introduction. Whilst doing this observe how the material of Sections 1-2 is used, so as to be better prepared for the problems that follow.

---

* This angle may have either sign.

** For nonplanar motions angular velocity is introduced in a more complicated way. In that context it also proves to be a vector quantity.

*** O_1, O_2 are any two (possibly, coincident) fixed points.

PROBLEM 1.* *Equilateral triangles ABC′, BCA′ and ACB′ are constructed on the sides of an arbitrary triangle ABC in the exterior of the triangle* (Fig. 33). *Prove that the centres O_1, O_2, and O_3 of these triangles are themselves the vertices of an equilateral triangle.*

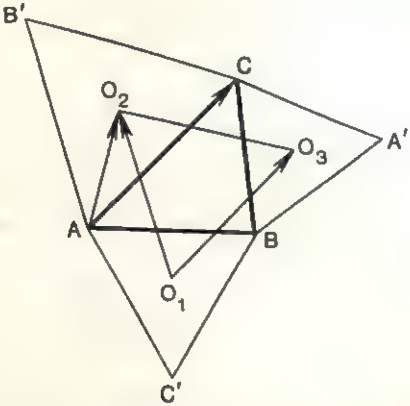


Fig. 33

SOLUTION. Fix the vertices A and B of the triangle ABC and move the vertex C. Let $\mathbf{v}_C$ be its velocity. The triangle ABC' will remain unaltered, and the vertices A' and B' of the equilateral triangles $A'BC$ and $AB'C$ will move in some definite manner. Consider the vectors $\overline{AC}$ and $\overline{AO_2}$. Evidently,

$$AO_2 = \frac{1}{\sqrt{3}} AC$$

In addition, the angle between the vectors $\overline{AC}$ and $\overline{AO_2}$ is $\pi/6$. Therefore, if we rotate the vector $\overline{AC}$ through the angle $\pi/6$ (its length remaining unaltered), and multiply the resulting vector by $1/\sqrt{3}$, then we have the vector $\overline{AO_2}$. This can be written as follows:

$$\overline{AO_2} = \frac{1}{\sqrt{3}} U_{\frac{\pi}{6}} \overline{AC}$$

* This problem, as well as most of those given below, has been borrowed from a series of "Mathematics Club Library" books by I. M. Yaglom and other writers. Some problems have been taken from J. Hadamard, Leçons de géométrie, vol. 1, 2ᵉ, éd., Paris, 1906.

By Theorem 8

$$\mathbf{v}_{O_2} = \frac{1}{\sqrt{3}} U_{\frac{\pi}{6}} \mathbf{v}_C$$

($\mathbf{v}_{O_2}$ being the velocity of the point O_2).

Similarly

$$\mathbf{v}_{O_3} = \frac{1}{\sqrt{3}} U_{-\frac{\pi}{6}} \mathbf{v}_C$$

Hence

$$\mathbf{v}_C = \sqrt{3}\, U_{\frac{\pi}{6}} \mathbf{v}_{O_3}$$

Consequently,

$$\mathbf{v}_{O_2} = \frac{1}{\sqrt{3}} U_{\frac{\pi}{6}} \sqrt{3}\, U_{\frac{\pi}{6}} \mathbf{v}_{O_3} = U_{\frac{\pi}{6}} U_{\frac{\pi}{6}} \mathbf{v}_{O_3} = U_{\frac{\pi}{3}} \mathbf{v}_{O_3} \qquad (1)$$

Now take the fixed point O_1 to be the pole. Then equation (1), by Theorem 8′, yields

$$\overline{O_1O_2} = U_{\frac{\pi}{3}} \overline{O_1O_3} + \mathbf{R}$$

where the vector $\mathbf{R} = \text{const}$, i. e. $\mathbf{R}$ is independent of the position of the moving point C. The vector $\mathbf{R}$ is unknown, but it is possible to find it by choosing any position of the point C. This position will be briefly referred to in what follows as the *determining position*. If it turns out that in the determining position of the point C the vector $\mathbf{R}$ is zero, then, as it is a constant, it must always be zero, i. e. we must always have

$$\overline{O_1O_2} = U_{\frac{\pi}{3}} \overline{O_1O_3} \qquad (2)$$

But this just means that the triangle $O_1O_2O_3$ is always equilateral! Equation (2) simply says that the segment O_1O_2 is obtained from the segment O_1O_3 by rotating it through an angle of $\pi/3$.

It remains to find a suitable determining position for the point C. It is advisable to choose the whole configuration to be as simple as possible. The configuration in this problem looks very simple (Fig. 34) if the point C is allowed to occupy a position such that the triangle ABC is equilateral. The symmetry takes place here: the configuration coincides with itself when rotated through an angle $\frac{2}{3}\pi$ about the centre of the triangle ABC. Therefore

the triangle $O_1O_2O_3$ turns out to be equilateral and, consequently,

$$\overline{O_1O_2} = U_{\frac{\pi}{3}}\overline{O_1O_3}$$

i.e. in this position $\mathbf{R} = \mathbf{0}$.

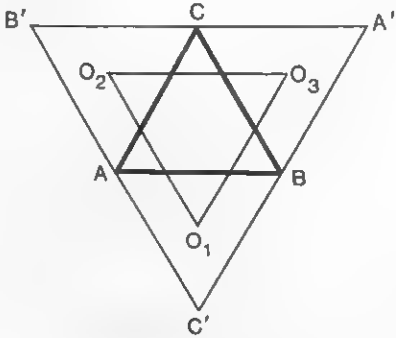


Fig. 34

EXERCISES. 1. Prove that the statement of Problem 1 is valid if the triangles ABC', BCA', ACB' are replaced by the triangles ABC'', BCA'', ACB'' symmetrical to the original triangles with respect to the sides of the triangle ABC (Fig. 35).

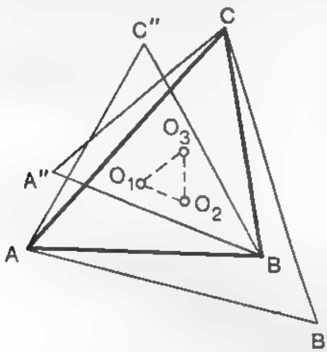


Fig. 35

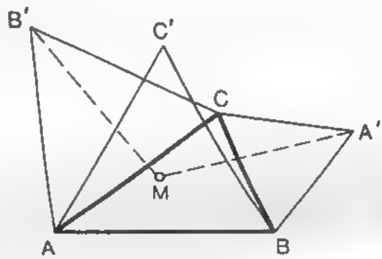


Fig. 36

2. Equilateral triangles BCA', ACB', ABC' are constructed on the sides of an arbitrary triangle ABC so that the vertices A' and A, B' and B are situated on different sides of BC and AC respectively, and C' and C on the same side of AB (Fig. 36).

Prove that if the point M is the centre of the triangle ABC', then the triangle $A'MB'$ is isosceles and the angle at its vertex M is $\frac{2}{3}\pi$.

3. Isosceles triangles BCA', ACB' and ABC' with angles at the vertices A', B' and C' equal to α, β and γ respectively, are constructed on the sides of an arbitrary triangle ABC in the exterior to the triangle (Fig. 37). Prove that if

$$\alpha + \beta + \gamma = 2\pi$$

then the angles of the triangle $A'B'C'$ are $\alpha/2$, $\beta/2$, $\gamma/2$, i.e. independent of the form of the triangle ABC. A particular case of this statement is known as the "Napoleon problem" [see the magazine "Kvant" (Quantum) (in Russian) 1972, No. 6, p. 26].

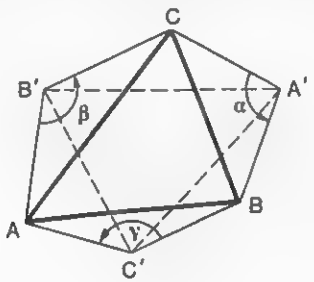


Fig. 37

PROBLEM 2. *A quadrangle ABCD is given. Isosceles right-angled triangles ABM, BCP, CDQ and DAS are constructed on its sides in the exterior of the quadrangle* (Fig. 38). *Prove that the segment MQ and SP are equal and perpendicular.*

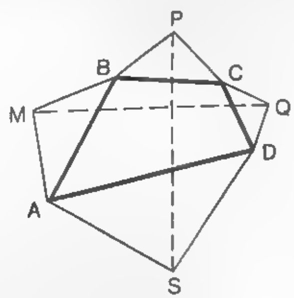


Fig. 38

SOLUTION. Fix the vertices A, B, and D and start moving the vertex C. Since

$$\overline{BP} = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \overline{BC}$$

we have

$$\mathbf{v}_P = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \mathbf{v}_C$$

Similarly

$$\mathbf{v}_Q = \frac{1}{\sqrt{2}} U_{-\frac{\pi}{4}} \mathbf{v}_C$$

Going over from the relation between the velocities to the relation between the radius vectors we get

$$\overline{SP} = U_{\frac{\pi}{2}} \overline{SQ} + \text{const}$$

Since $\overline{SQ} = \overline{SM} + \overline{MQ}$ and $\overline{SM} = \text{const}$, we have

$$\overline{SP} = U_{\frac{\pi}{2}} \overline{MQ} + \mathbf{R}$$

where $\mathbf{R} = \text{const}$.

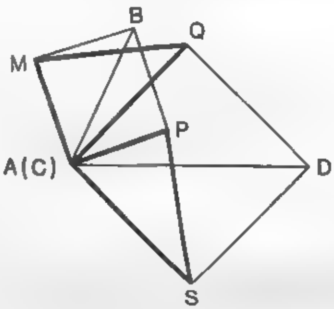


Fig. 39

Let us choose a determining position of the vertex C. For instance, let it coincide with the vertex A. The quadrangle $ABCD$ will then turn into two pairs of combined segments $AB = CB$ and $AD = CD$ (Fig. 39). The triangles ABM and CBP form a square constructed with AB as a diagonal. Similarly, the triangles

ADS and *CDQ* form a square with diagonal *AD*. It follows that by a rotation through $\pi/2$, the triangle *ASP* is brought into coincidence with the triangle *AQM* (the point *S* coinciding with the point *Q* and the point *P* coinciding with the point *M*). Therefore for the position of the point *C* under consideration

$$\overline{SP} = U_{\frac{\pi}{2}}\overline{MQ}$$

Thus in this case, and hence always, we have

$$\mathbf{R} = 0$$

i. e. it is always true that

$$\overline{SP} = U_{\frac{\pi}{2}}\overline{MQ}$$

But this just says that the segment *SP* can always be obtained from the segment *MQ* by a rotation through a right angle. Consequently,

$$SP = MQ, \quad SP \perp MQ$$

PROBLEM 3. *Squares are constructed on the sides of an arbitrary parallelogram ABCD exterior to the parallelogram. Prove that their centres M, P, Q, and S are themselves the vertices of a square* (Fig. 40).

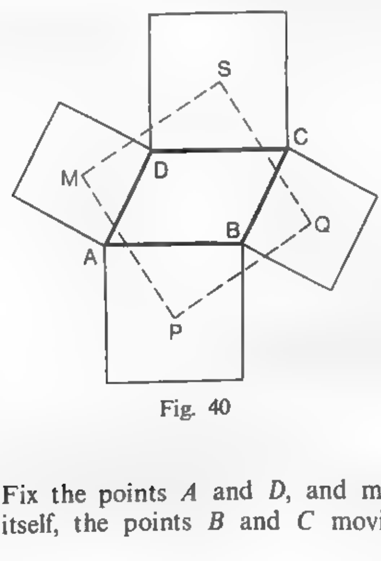


Fig. 40

SOLUTION. Fix the points *A* and *D*, and move the segment *BC* parallel to itself, the points *B* and *C* moving at the same

velocity. The point Q, the centre of the square* constructed on the segment BC, will also have this same velocity.

Calculate the velocity of the point S, the centre of the square constructed on the segment CD. Since

$$\overline{DS} = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \overline{DC}$$

we have

$$\mathbf{v}_S = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \mathbf{v}_C$$

Similarly,

$$\mathbf{v}_P = \frac{1}{\sqrt{2}} U_{-\frac{\pi}{4}} \mathbf{v}_B$$

Since

$$\mathbf{v}_B = \mathbf{v}_C = \mathbf{v}_Q$$

we have

$$\mathbf{v}_S = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \mathbf{v}_Q, \quad \mathbf{v}_P = \frac{1}{\sqrt{2}} U_{-\frac{\pi}{4}} \mathbf{v}_Q$$

Consequently,

$$\overline{MS} = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \overline{MQ} + \mathbf{R}_1, \quad \overline{MP} = \frac{1}{\sqrt{2}} U_{-\frac{\pi}{4}} \overline{MQ} + \mathbf{R}_2$$

where

$$\mathbf{R}_1 = \text{const}, \quad \mathbf{R}_2 = \text{const}.$$

Take as the determining position of the segment BC the position in which the quadrangle $ABCD$ is a square. Then it is found that

$$\mathbf{R}_1 = 0, \quad \mathbf{R}_2 = 0.$$

---

* The whole square will undergo what is said to be a translational movement.

Thus in this position, and hence also always,

$$\overline{MS} = \frac{1}{\sqrt{2}} U_{\frac{\pi}{4}} \overline{MQ}, \quad \overline{MP} = \frac{1}{\sqrt{2}} U_{-\frac{\pi}{4}} \overline{MQ}$$

and these equations imply that the quadrangle $MPQS$ is a square.

EXERCISES. 4. Prove that the statement of Problem 3 will continue to hold if all the squares are replaced by squares symmetrical to the original squares with respect to the sides of the parallelogram $ABCD$.

5. A quadrangle $ABCD$ is given. Prove that if the vertices P and S of the isosceles right-angled triangles ABP and CDS coincide, then so do the vertices of the isosceles right-angled triangles BCQ and DAT (Fig. 41). All the triangles are constructed in the interior of the quadrangle $ABCD$.

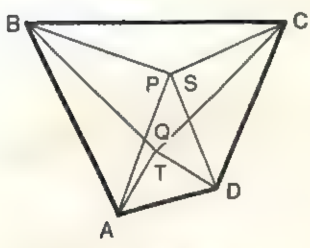


Fig. 41

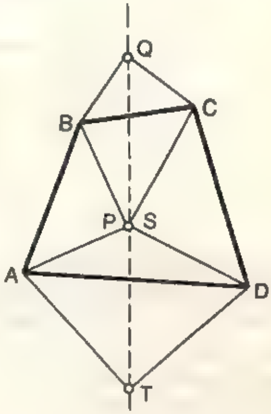


Fig. 42

6. A quadrangle $ABCD$ is given. Isosceles right-angled triangles ABP, BCQ, CDS, DAT are constructed on the sides BC and DA exterior to the quadrangle and on the sides AB and CD in the interior of the quadrangle (Fig. 42). Prove that if the vertices P and S coincide, then the segment QT passes through them and is bisected by them.

7. Prove that in Problem 1 the segments AA', BB', and CC' are equal and intersect in a single point, forming angles of $\frac{2}{3}\pi$.

PROBLEM 4. *Four straight lines a, b, c, and d are given intersecting pairwise at six points A, B, C, D, E, and F* (Fig. 43). *Prove that the midpoints M, P, and Q of the segments AC, BE, and DF lie in a straight line.*

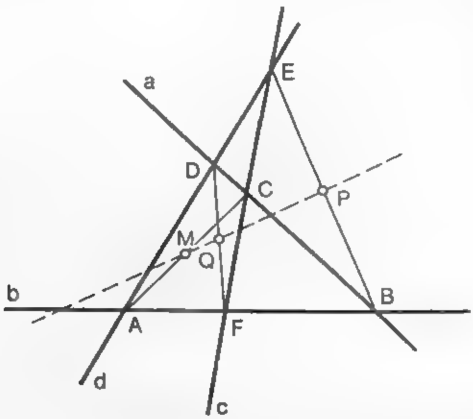


Fig. 43

SOLUTION. Fix the lines b, c, d and displace the line a parallel to itself. The points B, C, and D will then move along the lines b, c, and d. Denote their velocities by $\mathbf{v}_B$, $\mathbf{v}_C$, and $\mathbf{v}_D$.

Let a' be a displaced position of the line a, and let B', C', D' be corresponding displaced positions of the points B, C, and D (Fig. 44). The initial and the end points of the vectors $\overline{BB'}$, $\overline{CC'}$, $\overline{DD'}$ lie on the parallel lines a and a' respectively. Consequently, if the initial points of these vectors are allowed to coincide, then their end points will lie in a straight line parallel to a. The vectors $\overline{BB'}$, $\overline{CC'}$, and $\overline{DD'}$ are proportional to the velocities $\mathbf{v}_B$, $\mathbf{v}_C$, and $\mathbf{v}_D$ of the points B, C, and D. Therefore, if the vectors $\mathbf{v}_B$, $\mathbf{v}_C$, and $\mathbf{v}_D$ are measured from some point O, then their end points B_1, C_1, and D_1 will lie in a straight line (parallel to a). Hence, by the theorem of Subsection 1.7, there must exist constant numbers m and n such that

$$\mathbf{v}_B = \frac{m\mathbf{v}_C + n\mathbf{v}_D}{m+n} \tag{3}$$

The point M (see Fig. 43) is the midpoint of the segment AC,

i. e. $\overline{AM} = \frac{1}{2}\overline{AC}$. Since the point A is fixed, it follows that

$$\mathbf{v}_M = \frac{1}{2}\mathbf{v}_C \tag{4}$$

Similarly we get the equations

$$\mathbf{v}_Q = \frac{1}{2}\mathbf{v}_D, \quad \mathbf{v}_P = \frac{1}{2}\mathbf{v}_B \tag{5}$$

From equations (3) to (5) we get

$$\mathbf{v}_P = \frac{m\mathbf{v}_M + n\mathbf{v}_Q}{m+n}$$

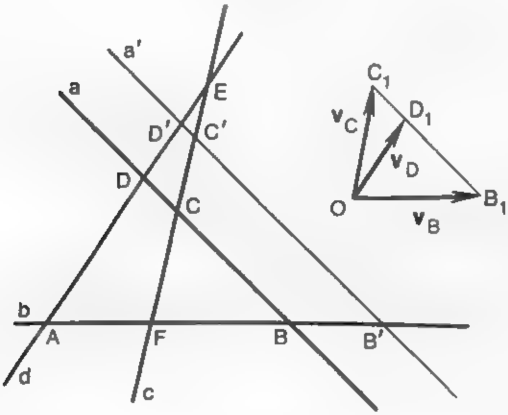


Fig. 44

Now take the fixed point E to be the pole. Then, applying successively Theorems 6′ and 7′, we get

$$\overline{EP} = \frac{m\overline{EM} + n\overline{EQ}}{m+n} + \mathbf{R}, \quad \mathbf{R} = \text{const} \tag{6}$$

In the determining position let the straight line a pass through the point E (Fig. 45). In this position the points D and C coincide with the point E. The points M, Q, and P are found to be the midpoints of the segments AE, FE, and BE. Since the points A, F, and B lie in a straight line, the points M, Q, and P will

also lie in a straight line (parallel to b). The triangles PEQ and B_1C_1O are similar (the sides of one being parallel to those of the other: $PE \parallel a \parallel B_1C_1$; $PQ \parallel b \parallel OB_1$; $EQ \parallel c \parallel OC_1$).

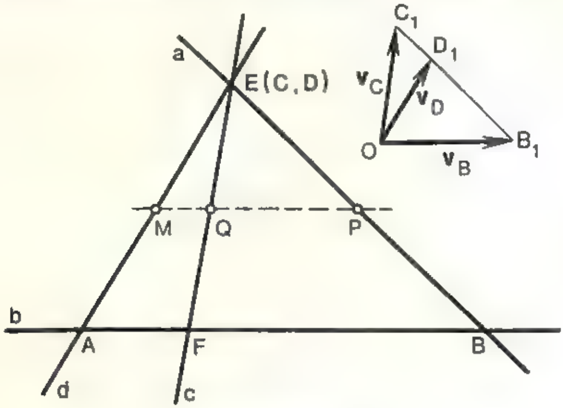


Fig. 45

In a similar manner the triangles PEM and B_1D_1O are similar. It follows from their similarity that

$$\frac{B_1D_1}{OB_1} = \frac{PE}{MP}, \quad \frac{C_1B_1}{OB_1} = \frac{PE}{PQ}$$

Hence

$$\frac{MP}{PQ} = \frac{C_1B_1}{B_1D_1}$$

i.e. the points P and B_1 divide the segments MQ and C_1D_1 respectively in the same ratio. But for the points B_1 this ratio is $m:n$. Consequently, the ratio is equal to $m:n$ for the point P as well. Thus

$$\overline{EP} = \frac{m\overline{EM} + n\overline{EQ}}{m+n} \tag{7}$$

Comparing equations (6) and (7), we see that in the determining position $\mathbf{R} = \mathbf{0}$. But since $\mathbf{R} = \text{const}$, we always have $\mathbf{R} = \mathbf{0}$. Hence equation (7) always holds and the points M, P, Q always lie in a straight line.

EXERCISE 8. Prove that the points of intersection of the altitudes of four triangles BCD, ABE, DEF, ACF formed by four pairwise intersecting lines a, b, c, d lie in a straight line (Fig. 46).

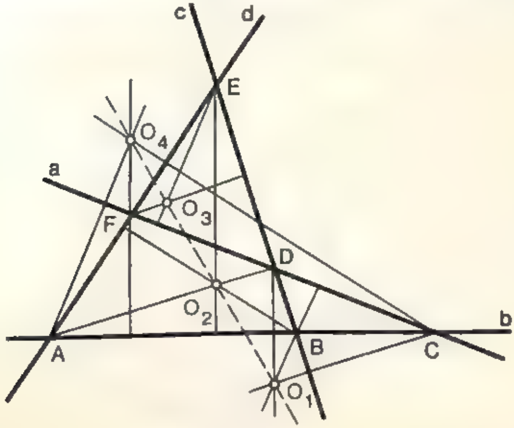


Fig. 46

PROBLEM 5. *Let a point P lie on the circumcircle K of a triangle ABC and let P_1, P_2, P_3 be the projections of the point P on the sides of $\triangle ABC$* (Fig. 47). *Prove that the points P_1, P_2, P_3 lie in a straight line (this is called the Simson line corresponding to the point P and the triangle ABC).*

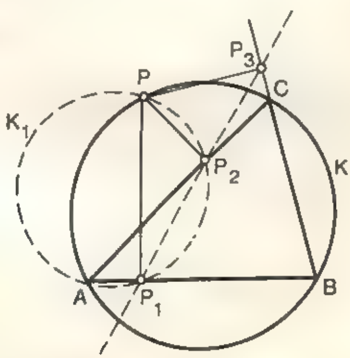


Fig. 47

SOLUTION. Rotate the sides AC and BC about the points A and B at the same angular velocity ω. The point C will then move round the circle* K. Since the angles PP_1A and PP_2A are right angles, the point P_2 moves round the circle K_1 passing through the fixed points A, P, P_1. Here the ray PP_2, which is always perpendicular to the ray AC, rotates about the point P at the

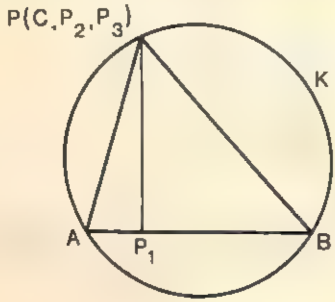


Fig. 48

same angular velocity ω (by Theorem 10 of Sec. 2). Since the rays PP_2 and P_1P_2 rotate so that their point of intersection P_2, moves round the circle K_1, the angle between them remains constant. Consequently, by Theorem 10 their angular velocities are equal. Therefore the angular velocity of the ray P_1P_2 is equal to ω. Similarly, the angular velocity of the ray P_1P_3 is equal to ω.

Thus the rays P_1P_2 and P_1P_3 rotate at the same angular velocity. Hence the angle between them is constant. In order to determine it to be zero (and thus complete the proof), consider the position in which the point C coincides with the point P (Fig. 48). In this position P_2 coincides with P_3 (and with P and C) so the angle under consideration is zero. Hence it is always zero.

EXERCISES. **9.** Let the points P and Q lie on the circumcircle K of a triangle ABC. Prove that the intersection point of the corresponding Simson lines p and q (Fig. 49) describes a circle K' when the point C moves round the circle K (the points A, B, P, and Q being considered fixed).

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* Since the angular velocities of the rays AC and BC are equal, by Theorem 10 the angle ACB always remains constant and, consequently, the point C moves along the circumference of the circle K.

10. Let a point P lie on the circumcircle K of a triangle ABC and let P_1, P_2, P_3 be points which together with the point P are symmetrical with respect to the sides of the triangle ABC. Prove that the points P_1, P_2, P_3 lie in a straight line passing through the point of intersection of the altitudes of the triangle ABC (Fig. 50).

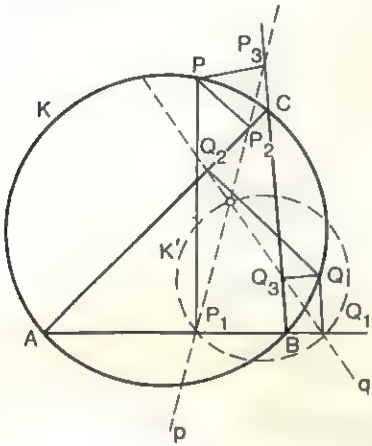

Fig. 49

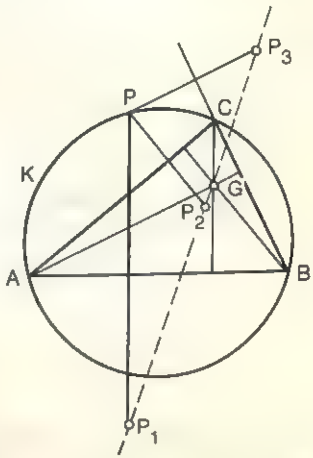

Fig. 50

PROBLEM 6. *Prove that the four circumcircles K_1, K_2, K_3, and K_4 of the four triangles ABD, BFC, CED, and AFE formed by four pairwise intersecting straight lines a, b, c, d pass through a point* (Fig. 51).

SOLUTION. Since the circles K_1 and K_2 have the point B in common, they have another point in common*. Denote this point by M. Prove that the point M also belongs to K_3 and K_4.

Fix the points B, C, and D and start rotating the straight lines b, c, and d about them at the same angular velocity ω. Since the angle BAD remains constant (see Theorem 10 of Sec. 2), the point of intersection of the straight lines b and d will move round the circle K_1. Similarly the point of intersection F of the

* If the circles K_1 and K_2 touched at the point B, then the triangles ABD and BFC would be similar. The straight lines AD and FC would then be parallel, which is contrary to the premises.

straight lines b and c will move round the circle K_2 and the point of intersection E of the straight lines c and d will move round the circle K_3.

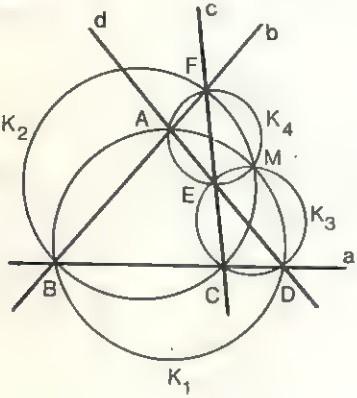

Fig. 51

At some instant the point A will coincide with the point M (see Fig. 52a) and hence the lines b and d will pass through M. Since M belongs to K_2 as well, and the straight line b intersects the straight line c on K_2 all three straight lines b, c, and d will pass through the point M at that instant. But the intersection point of the straight lines c and d lies on the circle K_3. Hence it follows that the circle K_3 passes through the point M.

To prove that the circle K_4 also passes through the point M, fix the points B, A, and F and rotate the straight lines a, d, and c about them at the same angular velocity (Fig. 52b). We can prove as in the proof above that at some instant all the three straight lines a, d, and c will pass through the point M. Hence the circle K_4 on which the straight lines c and d intersect also passes through the point M.

EXERCISES. **11.** An arbitrary point M is taken on the side AB of the triangle ABC. Prove that the centres O_1, O_2, and O_3 of the circumcircles of the triangles ABC, AMC, and BMC lie on a circle passing through the point C (Fig. 53).

12. Steiner's Theorem. Prove that the centres of the circles K_1, K_2, K_3, K_4 (see the condition of Problem 6) lie on the circle. This circle also passes through the point of intersection of the circles K_1, K_2, K_3, and K_4 (Fig. 54).

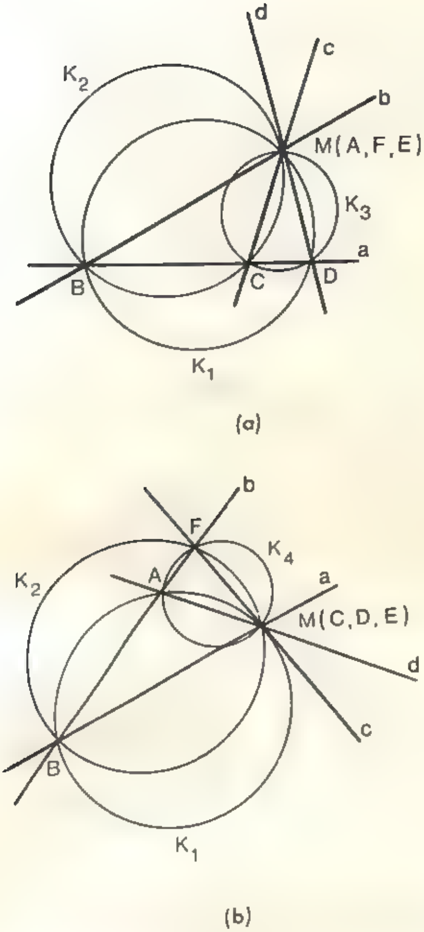

Fig. 52

PROBLEM 7. *Two circles K_1 and K_2* (Fig. 55) *intersecting in points A and B are given. A point M moving round the circle K_1 is joined to the points A and B. Let N and P be the intersection points of the straight lines MA and MB with the circle K_2. Prove that the centre O of the circumcircle K of the triangle MNP describes a circle.*

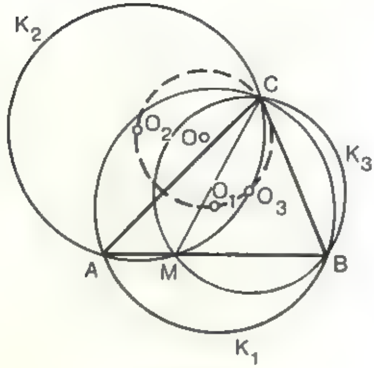

Fig. 53

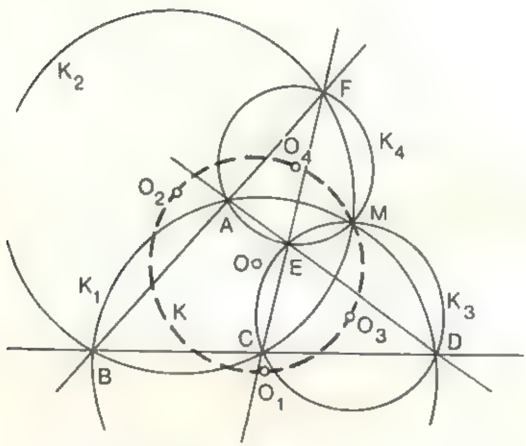

Fig. 54

SOLUTION. When the point M moves round the circle K_1, the rays AN and BP rotate about the points A and B at the same angular velocity ω. The angular velocities of the radii O_2N and O_2P drawn from the centre O_2 of the circle K_2 to the points N and P are equal* to 2ω. It follows that the angle PO_2N

* Indeed, suppose the ray AN rotates through some angle NAN' in the time interval $[t, t + \Delta t]$. In the same time interval the radius O_2N will rotate through the angle NO_2N' which, being an angle at the centre, is equal to twice the inscribed angle NAN'. Since the relation $\angle NO_2N' = 2\angle NAN'$ holds for any Δt, the angular velocity of the radius O_2N is equal to twice the angular velocity of the ray AN.

remains constant and the triangle PO_2N moves remaining unaltered. Since the length of the chord PN and the angle PMN are constant, the circumcircle K of the triangle MNP moves remaining unaltered, as also does the triangle PON together with the centre O and the chord PN of the circle K. It follows that the triangle

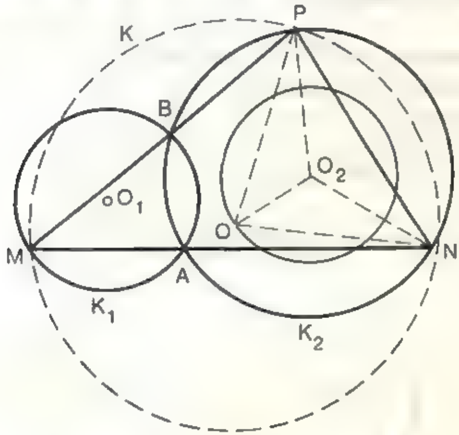

Fig. 55

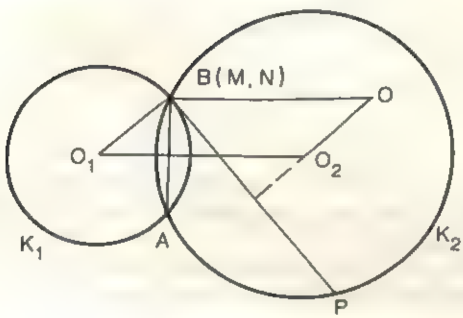

Fig. 56

O_2ON also moves remaining unaltered. Since its vertex O_2 is fixed, the point O describes a circle.

We show that the radius of the circle is equal to the radius of the circle K_1. To do this, let the point M coincide with the point B (Fig. 56). The secant MBP will then become the tangent

to the circle K_1 at the point B (see Subsection 2.2). The chord MA will coincide with the chord AB and the point N will coincide with the point B. The triangle MNP will "degenerate" into the segment BP (covered twice). The centre O of the circumcircle of the triangle lies at the point of intersection of the perpendicular to the chord BP drawn through its midpoint and the perpendicular to the chord AB drawn through the point B*. It follows that the quadrangle O_1O_2OB is a parallelogram and that $O_2O = R_1$.

Thus the point O describes a circle with centre at the point O_2 and radius R_1.

EXERCISES. 13. Prove that the side PN of the triangle MNP (see the statement of Problem 7) is tangent to some fixed circle.

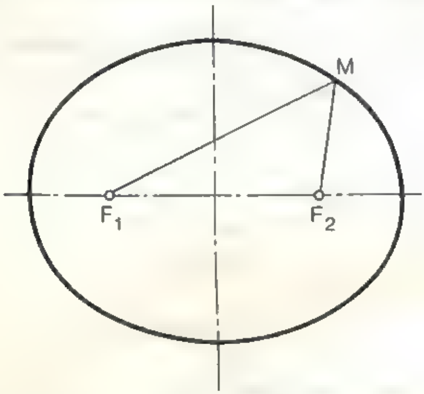

Fig. 57

14. Prove that the point of intersection of the altitudes of the triangle MNP constructed in Problem 7 describes a circle when the point M moves.

In conclusion we shall consider some properties of the ellipse, hyperbola, and parabola. Definitions of these curves will be given below.

An *ellipse* is a curve consisting of all points for which the sum of the distances from them to two given points, F_1 and F_2, is equal to a given constant (Fig. 57). The points F_1 and F_2 are called the *foci* of the ellipse.

* The side MN of the triangle MNP has degenerated into a point, but the direction of this degenerate side is determined when proceeding to the limit, and it coincides with that of the chord AB.

PROBLEM 8. *Prove that a tangent to an ellipse makes equal angles with the radius vectors drawn from the foci to the point of tangency.* Conversely, if the tangent to any point of a curve makes at each point equal angles with the radius vectors drawn from two fixed points, F_1 and F_2, to the points of tangency, then the curve is an ellipse with its foci at the points F_1 and F_2 (or an arc of this ellipse).

SOLUTION. Let a point M move round an ellipse at a velocity $\mathbf{v}$. The projections of the vector $\mathbf{v}$ on the radius vectors* $\mathbf{r}_1 = \overline{F_1M}$ and $\mathbf{r}_2 = \overline{F_2M}$ (Fig. 58) are, respectively,

$$v_1 = pr_{\mathbf{r}_1}\mathbf{v} = -v\cos\alpha, \quad v_2 = pr_{\mathbf{r}_2}\mathbf{v} = v\cos\beta \tag{8}$$

where α and β are the angles between $\mathbf{r}_1$, $\mathbf{r}_2$ and the tangent. Since by the definition of the ellipse $|\mathbf{r}_1| + |\mathbf{r}_2| = \text{const}$, we have**

$$v_1 + v_2 = 0$$

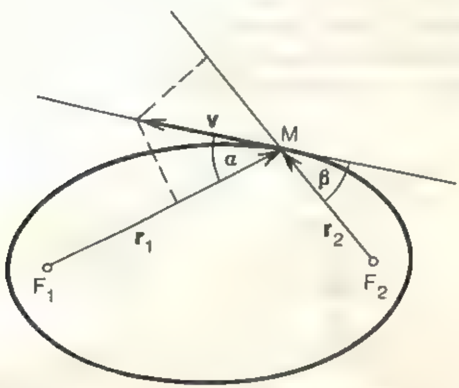

Fig. 58

Substituting the expressions for v_1 and v_2 from (8) into this equation we have:

$$v\cos\alpha - v\cos\beta = 0$$

* That is on the axes directed along these vectors.

** Since the sum $|\mathbf{r}_1| + |\mathbf{r}_2|$ is a constant, the increments of the lengths of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are equal in magnitude and opposite in sign. Consequently, the rates of change of the lengths of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are also equal in magnitude and opposite in sign. According to 2.5 these rates are precisely v_1 and v_2.

or

$$\cos\alpha = \cos\beta$$

whence $\alpha = \beta$ since α and β are acute angles.

Conversely, let a tangent to a curve L make equal angles with the radius vectors drawn from the fixed points F_1 and F_2 to the point of tangency. Projecting the velocity $\mathbf{v}$ of a point moving along the curve L onto the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of that point we have

$$v_1 = pr_{\mathbf{r}_1}\mathbf{v} = -v\cos\alpha, \quad v_2 = pr_{\mathbf{r}_2}\mathbf{v} = v\cos\alpha$$

where α is the angle between the tangent and the radius vectors. Adding these equations we have

$$v_1 + v_2 = 0$$

from which it follows that the sum of the lengths of the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is a constant, i. e. the curve L is an ellipse.

A *hyperbola* is a curve consisting of all points for which the difference between the distances from two given points, F_1 and F_2, called the foci, is a constant (Fig. 59).

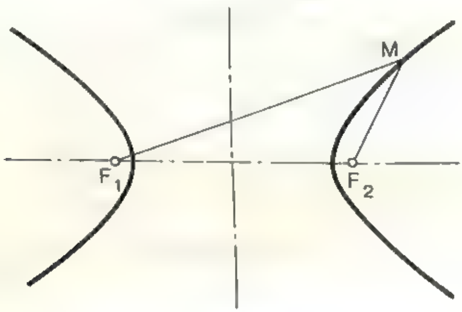

Fig. 59

EXERCISE 15. Prove that a tangent to a hyperbola bisects the angle between the radius vectors drawn from the foci to the point of tangency (Fig. 60).

Conversely, if a tangent to a curve is at each point the bisector of the angle made by the radius vectors drawn from two fixed points, F_1 and F_2, to the point of tangency, then the curve L is a hyperbola with foci at the points F_1 and F_2 (or an arc of that hyperbola).

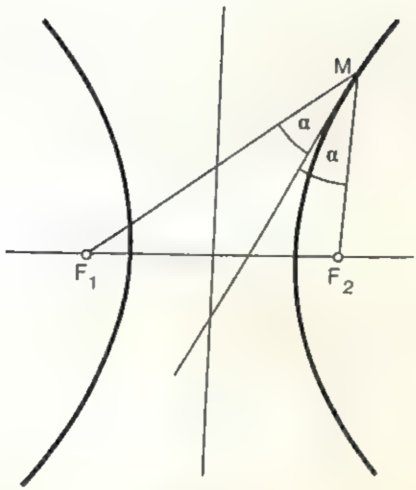

Fig. 60

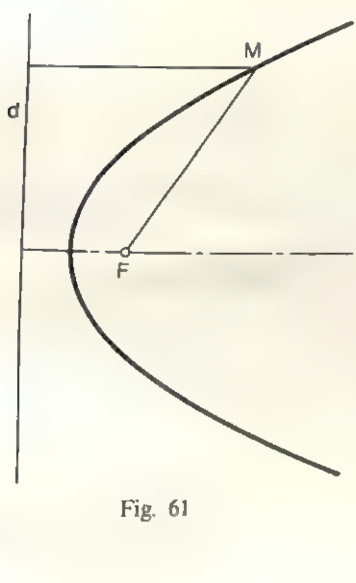

Fig. 61

A *parabola* is a curve consisting of all points for which the distances from a given point F, called the focus, and a given straight line d, called the directrix, are equal (Fig. 61).

EXERCISE **16.** Prove that a tangent to a parabola is the bisector of the angle between the radius vector drawn from the focus F to the point of tangency and the perpendicular dropped from the point of tangency to the directrix d (Fig. 62).

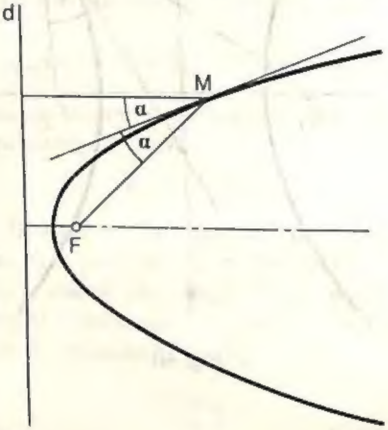

Fig. 62

Conversely, let a tangent to a curve be at each point the bisector of the angle made by the radius vector drawn from a fixed point F to the point of tangency and the perpendicular dropped from the point of tangency to a fixed straight line d. Then the curve is a parabola with focus F and directrix parallel to d (or an arc of that parabola).

For additional exercises the reader is recommended to solve by the kinematic method the following problems from the magazine "Kvant": 1) 1970, No. 4, p. 27, Problem M18 (a); 2) 1971, No. 4, p. 33, Problem M79; 3) 1973, No. 4, p. 43, Problem M198; 4) 1974, No. 11, p. 40, Problem M291; 5) 1974, No. 12, p. 44, Problem M297.

HINTS ON THE EXERCISES

1. The solution is similar to that of Problem 1. It should be observed that for the equilateral triangle ABC the points O_1, O_2, O_3 will now coincide (the triangle $O_1O_2O_3$ will "degenerate" to a point), and the vectors $\overline{O_1O_2}$ and $\overline{O_1O_3}$ will vanish. Using this we derive from the equation

$$\overline{O_1O_3} = U_{\frac{\pi}{3}}\overline{O_1O_2} + \mathbf{R}$$

that $\mathbf{R} = 0$.

2. Fix the points A, B and move the point C. While doing this observe the velocities of the points A' and B'. In the determining position let the point C coincide with the point C'.

3. The solution is similar to that of Problem 1. In the determining position bring the point C to coincide with one of the points, A or B.

4. The solution is similar to that of Problem 3.

5. Fix the points A and B and move the points C and D. In doing this the movement of the points C and D must be coordinated so that the triangle CSD with the fixed vertex S remain isosceles. Show that $\mathbf{v}_Q = \mathbf{v}_T$.

6. In a manner similar to the preceding exercise show that $\mathbf{v}_Q = -\mathbf{v}_T$.

7. Prove in a manner similar to the solution of Problem 1 that

$$\overline{AA'} = U_{-\frac{\pi}{3}}\overline{C'C}, \quad \overline{BB'} = U_{\frac{\pi}{3}}\overline{C'C}$$

From this deduce that the straight lines AA', BB', CC' intersect pairwise on the circumcircle of the triangle ABC.

8. Fix the straight lines b, c, and d and move the straight line a parallel to itself at a constant velocity. Further prove that

$$\mathbf{v}_{O3} = \lambda\mathbf{v}_{O1}, \quad \mathbf{v}_{O4} = \mu\mathbf{v}_{O1}$$

where $\lambda = \text{const}$, $\mu = \text{const}$. Establish that the constants $\mathbf{R}_1$ and $\mathbf{R}_2$ are zero by considering two (!) determining positions: a passes through the point A, a passes through the point B. Make use of the fact that a vector simultaneously collinear with two intersecting straight lines must be the zero vector.

9. When the point C moves the angular velocity of rotation of the Simson lines p and q equals that of the rays AC and BC.

10. Using the result of Problem 5, first prove that the points P_1, P_2, P_3 lie in a straight line. Then, having fixed the points

A, B, and P, rotate the straight lines AC and BC round the points A and B at the same angular velocity. Consider the angular velocities of the straight lines $P_1P_2P_3$ and P_1G. Choose a determining position in the same way as in Problem 5.

11. Fix the points A and C and rotate the straight lines AB, CM and CB at the same angular velocity. Observe the movement of the points O_1, O_2, and O_3. Consider the position in which the straight lines AC and AB coincide.

12. Fix the points B, C, and D and rotate the straight lines BF, CF, and DA at the same angular velocity until they pass through the point M. Then use the result of the previous exercise.

13. Make use of the fact that the length of the chord PN remains constant (see the solution of Problem 7).

14. Prove as a preliminary that the distance from a vertex of the triangle to the intersection point of the altitudes is equal to twice the distance from the centre of the circumcircle of the triangle to the corresponding side. This is easy to do without kinematics. Then make use of the fact that when the points M, N, and P move the straight lines O_1M and NP remain perpendicular to each other and the distance from the point O to the straight line NP remains constant.

15. The solution is similar to that of Problem 8.

16. The solution is similar to that of Problem 8, but instead of the rate of change of the length of the second radius vector one should consider the rate of change of the distance of the moving point from the directrix.

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When solving a geometrical problem it is helpful to imagine what would happen to the elements of the figure under consideration if some of its points started moving. The relationships between various geometrical objects may then become clear graphically and the solution of the problem may become obvious.

The relationships between the magnitudes of segments, angles and so on in geometrical figures are usually more complicated than the relationships between their rates of change when the figure is deformed. Therefore, in solving geometrical problems one may benefit from a "theory of velocities", i.e. from kinematics.

This little book uses a number of examples to show how kinematics can be applied to problems of elementary geometry, and gives some problems for independent solution. The necessary background information from kinematics and vector algebra is given as a preliminary.

The book is based on lectures given by the authors for school mathematics clubs at the Kharkov State University named after A. M. Gorky. It is intended for high school students.

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